PROOF OF CHURCH-ROSSER THEOREM IN CALCULUS OF CONSTRUCTIONS

A thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

by

AROURI NARAYANA

to the

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

APRIL 1991

ACKNOWLEDGEMENTS

First of all, I would like to thank my thesis supervisor Dr.Somenath Biswas for not just teaching the subject but for being a model in all spheres of life.

I thank Durga Ramesh, Peyyeti Sharma and Kordale Rambo for their company and affection which kept me living the life. I won't be able to forget the association of the Telugu Sangha which filled the jollier part of my stay here. Especially, I would like to thank the company of Ravi Chander, DVSSM, Ravi Shaker, Baddu, Bendapudi Perraju, BV Rao and all others.

I would like to thank Kalyan Shanker Basu, Gerard Huet, Christine Pauline Mohring and Mauny for their patience and interest in clarifying my doubts.

CERTIFICATE

This is to certify that the work entitled "PROOF OF CHURCH-ROSSER THEOREM IN CALCULUS OF CONSTRUCTIONS" by Aroori Narayana has been carried out under my supervision and has not been submitted elsewhere for a degree.

S. Biswas

(Somenath Biswas)
Department of Computer Science and Engg'.
Indian Institute of Technology
KANPUR

Kanpur
April 1991.

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Doc. No 112193

CSE 1991-M-NAR-PRO

ABSTRACT

In this thesis the Calculus of Constructions is used to formalise a constructive proof of Church-Rosser theorem for λ-calculus The original proof of Martin-Löf and Tait is used for the present purpose The proof is formalised for λ-calculus with de Bruijn notation in which the variable occurrence is replaced by a number denoting the depth of the occurrence from the binding λ This notation remove the need for the α-reduction The proof consists in showing that a special reduction called *walks* in which the reductions are performed in an inside first manner allows the diamond property and then showing that the diamond property of *walks* relation implies the diamond property of the usual reduction

CONTENTS

CHAPTER 1
INTRODUCTION

In the current thesis we shall use the Calculus of Constructions to provide a constructive proof of Church-Rosser theorem for λ-calculus. The Church-Rosser theorem states that if a λ-term X reduces to λ-terms Y and Z then there exists a λ-term W such that Y and Z reduce to it. The Church-Rosser theorem assures the uniqueness of the normal form. The proof is formalised for λ calculus with de Bruijn notation. The formalised proof script is presented in the appendix.

The λ-calculus was originally conceived by Church as part of a general theory of functions and logic intended as a foundation for mathematics. Although the full system turned out to be inconsistent the subsystem dealing only with functions turned out to be a successful model of computation. Instead of trying to model the logic formulae by pure λ-terms which gave Church an inconsistent system, one can model the proof terms of logic by typed λ-terms which gives a consistent system. *Calculus of Constructions* (abbreviated COC hereafter) [CH88] follows this paradigm and has emerged as an elegant presentation of constructive logic. The proof theoretic strength of the system is not yet completely understood. The inherent structure of the λ calculus for this purpose is higher order impredicative λ-calculus with dependent types.

The COC is interesting to computer science since it presents the *proofs-as-programs* paradigm in a clear light. When we identify the types with the propositions the resulting proof terms can be seen as the programs which realise their types in Kleene's sense of realisability for intuitionistic logic.

Church-Rosser theorem is a celebrated mathematical result about λ calculus It implies the consistency of the λ-calculus as a rewrite rule system It has taken more than thirty years for the evolution of a clear proof of Church Rosser Most of the early proofs are either incomplete or defective The current study presents a machine checking of the proof by Martin-Lof/Tait

The Martin-Lof s proof for λ-calculus with standard notation is presented in the appendix of [HS72] In the current study the λ-calculus is formulated in de Bruijn notation This notation assumes importance in the machine implementations of λ-calculus The de Bruin notation does away with the α-reduction But the proof gets more complex with the properties of de Bruijn indices The whole proof as presented in [HS72] is tailored for de Bruijn indice and the proof is machine checked in COC The machine checked proof script is presented in the appendix B The implementation of COC used for the current study is COC V 4 10 developed at INRIA

A mechanical proof for Church-Rosser in Boyer-Moore logic is presented in [Sh88] The Boyer-Moore logic is completely different from COC in content and philosophy The Boyer-Moore s is a quantifier free first order logic The language is a form of Pure-Lisp The terms of Boyer-Moore are either variables or of the form (FN t_1 t_n) where FN is an n ary function and t_1 t_n are terms The COC is an higher order impredicative intuitionistic logic

The following sections of the current chapter present a brief survey of literature on COC The second chapter presents an informal proof on the lines of [HS72] Chapter three is a rigorous formulation of the proof with de Bruijn notation This chapter can also be seen as an English translation of the machine checked proof script which is added as appendix B The chapter 4

briefly summarises our experience with the system of COC Appendix A is an example session of the system V 4 10

1 COC from λ-calculus point of view

COC can also be viewed in terms of Barendregt *General sed Type System (GTS)* [BH90] which clearly shows the place of this formalism amongst such ones This view also show that COC is the most general amongst such systems

In this section we present COC from the λ-calculus point of view We describe Barendregt s *General sed Type System* (GTS) a single uniform presentation of the forest of λ calculus systems and show the place where the COC nicely fits in GTS

1 1 Definition (i)(term of GTS) The set T of terms of a GTS is defined by

I= V C | T T λV T T ΠV T T

where C is the set of constants and V is the set of variables

(ii) (statement) A statement of a GTS is of the form M A with A∈T M is called the term and A is called its type

(iii) A context is a finite linearly ordered sequence of statements with distinct variables as terms

1 2 Definition : A specification of a GTS is a triple=(S A R) such that

-S⊆C the elements of S are called sorts

-A is the set of statements of the form c s with c∈C and s∈S The elements of A are called axioms

-R is the set of pairs of th form (s1 s2) with s1 and s2∈S The elements of R are called rules

1.3 Definition (GTS) : Given a specification of a GTS G=(S A R) the corresponding λ-calculus system derives the statements relative to a context Γ. The rules (s1 s2)∈R determine which abstractions are allowed.

(Axiom) $\quad \vdash c\ s \qquad \text{if } (c\ s)\in A$

(Start) $\quad \dfrac{\Gamma \vdash A:s}{\Gamma,\ x:A \vdash x:A} \quad \text{if } s\in S \text{ and } x \text{ is fresh}$

(Weakening) $\quad \dfrac{\Gamma \vdash B:C \qquad \Gamma \vdash A\ s}{\Gamma,\ x:A \vdash B\ C} \quad \text{if } s\in S \text{ and } x \text{ is fresh}$

(Π-elimination) $\quad \dfrac{\Gamma \vdash F:(\Pi\ x:A\ B) \qquad \Gamma \vdash a:A}{\Gamma \vdash (F\ a)\ [a/x]B}$

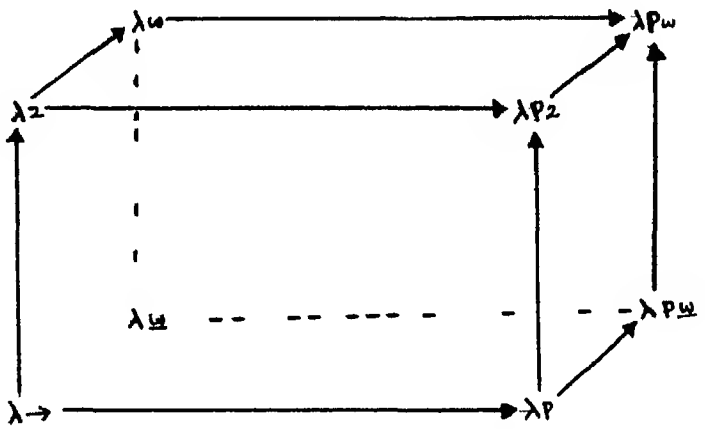
(Π-formation) $\quad \dfrac{\Gamma \vdash A:s1 \qquad \Gamma,\ x:A \vdash B\ s2}{\Gamma \vdash (\Pi\ x:A\ B) : s2} \quad \text{if } (s1\ s2)\in R$

(Π-introduction)

$$\frac{\Gamma \vdash A\ s1 \qquad \Gamma,\ x:A \vdash B:s2 \qquad \Gamma,\ x:A \vdash b\ B}{\Gamma \vdash (\lambda\ x:A\ b)\ (\Pi x:A\ B)} \quad \text{if } (s1\ s2)\in R$$

(Conversion) $\quad \dfrac{\Gamma \vdash A\ B \qquad \Gamma \vdash B\ s}{\Gamma \vdash A\ B} \quad \text{if } s\in S \text{ and } B \equiv_\beta B$

For Various GTS specifications we get different λ-calculus systems The calculus of constructions is a λ-calculus system with sorts S= * □} a ioms A= (*:□)} and rules R= (* *) (* □) (□ *) (□ □)} The rule (* *)∈R gives the system the elements dependent on elements (□ *)∈R gives the polymorphic types (* □)∈R gives the types dependent on elements and (□ □)∈R gives the types dependent on types The COC is thus an impredicative higher order typed λ-calculus with dependent types Without the rule (* □)∈R we get the system Fω of Girard Thus the COC is a conservative e tension of system Fω The relation of COC with seven other common λ-calculus systems can be shown as a cube called λ-cube



λ-cube

For all the eight systems the sorts and a ioms are common

i e ⊃= * □ and A= (*:□) The rules are as follows

(1) λ (simply typed λ calculus) R= (* *)

(2) λ2 (System F) R= (* *) (□ *)

(3) λ$\underline{\omega}$ R= (* *) (□ □)

(4) λω (System Fω) R= (* *) (□ *) (□ □)}

(⊃) λP (LF) R= (* *) (* □)

(6) λP2 R=[(* *) (□ *) (* □)

(7) λP$\underline{\omega}$ R= (* *) (□ □) (* □)}

(8) λPω (COC) R= (* *) (□ *) (□ □) (* □)

Thus the COC or λPω is the most general among the possible systems The system λP is given that name because predicate logic is interpreted in it The approach followed for this interpretation is called *propositions-as-types* paradigm and is the basis of several languages like AUTOMATH [dB80] NUPRL [Co56] LF [HHP87]

In all the above systems the terms of * are called *types* and the terms of □ are called *kinds*

2 COC from logic point of view

In the previous section we have seen that COC is an impredicative higher-order λ calculus with dependent types By identifying * with Prop the set of all logic propositions we get an impredicative higher order minimal intuitionistic logic This is the pure Calculus of Construction as presented in [CH88] Here we try to model the syntactic behavior of logic by λ calculus β-reduction In this framework the types are identified with the formulae and the λ-terms with the proofs Instead of directly identifying the types with the propositions another way of viewing this is the type corresponding to a proposition is the type of proofs of the proposition The logic thus becomes the expressions of the fact that a proposition is true if and only

if its corresponding type of proof is inhabited

In [Coq89] Coquand call the types of previous section a small types and kinds of previous section as big type Only the small types correspond to the formulae of logic The big types are the types of formulae and higher order predicates Both the small and big types can be used as individual types We have both the calculus of functions (when we consider types as individual types) and calculus of logic (when we consider the types a the types of proofs) in a single frame-work Of course there is some confusion of understanding between the two

We have terms of 4 levels The terms of a level are the types of the terms of a lower level

The terms of the 0 th level are proof terms The terms of level 1 are the small types or the formulae of logic Only these are called types by Barendregt[BH90] The terms of level 2 are the type of terms of level 1 and are called big types by Coquand[Coq89] and kinds by Barendregt[BH90] Here we are able to form A Prop Prop- Prop etc The terms of level 3 are singleton This is □ of [BH90] and Type of [Coq89] This is the type of all types

We are able to form four types of products:

(i) Product formation from small types to small types give rise to the concept of elements depending on elements eg If A and B are of type * the type A- B is also of type * This is arrow type in λ-calculus and implication in logic

(ii) The product formation from big types to small types give rise to the effect of polymorphism of the λ calculus and second order quantification of logic For eg A * |- (A:*)(A- A) : * The type (A:*)(A- A) is the type of polymorphic identity in λ-calculus When we identify * with Prop in logic we have the formula (A Prop)(A= A)of type Prop It is here we get an

impredicative system The concept of defining an element of a set by using the description of the same set is called *impredicativism* Here we are defining an element of Prop by quantifying over Prop itself The actual implementation of COC V4 10 done by Huet [H89] contains an infinite number of hierarchies All the levels beyond level one are predicative while the level one is impredicative The impredicativity at two levels gives rise to inconsistency to the system as has been observed by Girard [Coq86] for system U

(iii) The product formation from small type to a big type gives the effect of element dependent types For eg A * |- (: A)* □ which is the type A- Prop the type of predicates over A

(iv) The product formation from a big type to big type gives the ability to use higher order quantification For eg A * B * |- λ(A Prop) λ(B Prop) (C Prop)(A- B C)- C This is the connective AND which is of the type Prop- Prop- Prop The connective AND is generic in the sense that we can plug in any two propositions

3 Implementation of COC

The implementation of COC which is used in the current study is the version 4 10 written by Gerard Huet [H89] developed at INRIA France This section briefly describes the proof description language of this implementation called Mathematical vernacular developed by Gilles Dowek [D89] the facility for inductive definitions developed by Pauline [P89] and an automatic theorem prover called Syntthesis developed by Coquand [Coq89S]

3 1 Mathematical Vernacular :

The purpose of the proof description language is to make the language of COC as close as possible to the language in which the mathematical books are written Each and every vernacular

tatement is translated into the basic commands of the constructive engine [H89] of COC The fundamental vernacular which is mainly made use of in the machine proof of Church-Rosser is explained below

In vernacular a variable is declared by

Parameter <name of variable> <type>

The syntax for adding an axiom is

Axiom <name> <statement>

The syntax for proving a new theorem is

Theorem <name of theorem> <statement>

Proof proof of the theorem>

New definition is added by the syntax

Definition <name>=<body>

Local variables and hypotheses are added respectively by

Variable <name> <type> and

Hypothesis <name> <statement>

There is a provision for splitting the proof into *Sections Subsections* and *Chapters* with the usual scoping rules

3 2 Inductive definitions :

Inductive definitions are the means of giving complete specification of a notion from the description of their constructors A macro command is available to the user which generates the type introduction and elimination rules from the specification For eg the definition of sum type is given as

Inductive Definition sum [A B Prop] Prop

= left A->(sum A B)

| right B->(sum A B)

The system generates the following

) = (C *)(A- C) (B- C)->C

= [H:(sum A B)]H

left = (Ea AJEC *JEf:A- C3E, B- C3(f a)) A (sum A B)

right = (Eb B3EC *JEf A- C3L, B C3(, b)) B- (sum A B)

Two specifications in two different constructors introduce a sum type between them and in the same constructor introduce a product type between them

3 3 Tactics

The tactics theorem prover called Synthesis machine is developed by Coquand [Coq89a] The philosophy of the tactics is the goal directed proving which is the reverse of the natural deduction The tactic is a CAML function which when applied to a goal reduces it to simpler subgoal A tactical is a higher order CAML function for combining tactics to build new tactics Different tactics are provided for different inference rules of COC The tactic which were found to be much useful in the construction of the proof of Church Rosser are

(a) assumption tactic : This takes the proof of a goal from the hypotheses conte t This tactic corresponds to rule *start*

(b) intro tactic _ This introduces the abstraction in the proof This corresponds to Π-*introduction* rule

(c) resolve tactic : This introduces an application in the proof structure The effect corresponds to Π-*elimination* rule

(d) change tactic _ The effect of this tactic corresponds to *conversion* rule

The basic tacticals which were found to be useful are

tac1 THEN tac2

This applies the tac2 tactic to all the new goals generated by the tac1 tactic

REPEAT tac

This repeats the tactic *tac* till it fail

A compound tactic caled *automatic* built from the above tactics was found to be very much useful This performs a series of five successful resolutions with the lemmas added to add_resolve_list interspersed with assumptions and fails if it can not complete the current goal

Chapter 2

Informal Proof

1 Introduction to λ-calculus and de Bruijn notation

The λ-calculus is a type free theory about the functions as rule rather than graphs It studies the problems such as what all can be computed and which computable processes are equivalent These computable processes are called λ terms of the λ-calculus In our presentation the main relation between the λ-terms is reduction which is sufficient for the proof of Church-Rosser

We begin by assuming that there is an infinite set V of variable and a finite or infinite set C of constants An atom is a constant or a variable

1 1 Definition (λ-terms) The set of λ-terms Λ is defined by

(i) Every atom is a λ term

(ii) If X and Y are λ-terms then (X Y) is a λ-term

(iii) If Y is a term and x is a variable then λ Y is a λ-term

Examples of λ-terms are (λx (λy (y))) ((λx x)(λ (y)))

1 2 Definition (Free and Bound variables) An occurrence of a variable in Y is bound iff it is inside a part of Y of the form λx otherwise it is free

Substitution is the process of replacing all the free occurrences of a variable in a term by another term

1 3 Definition (Substitution) For any terms M N and any variable the result [N/x]M of substituting N for every free occurrence

of < in M is defined as follows by induction on the construction of M

(i) [N/ I] = N

(ii) [N/ Iy] = y for all atoms y≠

(iii) [N/ I(M1 M2)] = ([N/ IM1][N/ IM2])

(iv) [N/ I(λy M)] = λy M if =y

= λy [N/ IM] otherwise

To ensure that the substitution has the intended meaning no free variable occurrence in N can be allowed to become bound in [N/<IM] For this we rename the bound variable as and when needed Two terms which are equivalent except for the renaming of bound variables are called α-reducible to each other

1 4 Definition (α-reduction) A term X goes to a term Y in a α-step iff X and Y are identical except that the names used to denote the occurrence of variables bound by a λ in X and the corresponding in Y might differ uniformly

The computation is actually done by a rule known as β-step A subterm of the form (λ: M)N is called a redex

1 5 Definition (reduction) A term X goes to Y in a β-step iff a subterm (λx M)N of X is replaced by [N/ IM] to obtain the term Y A term X reduces to a term Y iff Y is obtained from X by a finite (perhaps empty) series of β-steps and α-steps This is denoted as X ↠ Y

Hence β-reduce relation is the reflexive transitive closure of β-step relation

eg (λ (λy (y))) v ↠ [v/ I(λy (y)) = (λy (y v))

1 6 de Bruijn Notation

With the previous notation for λ-terms we have to take care

of renaming of bound variable in the reduction process at every β-step. With the notation developed by de Bruijn in [dB72] for the implementation of Automath two terms which are α equal are always syntactically identical. In this notation a variable occurrence in a λ-term is represented by an integer denoting the depth from the binding λ. We also do away with the tagging of name of variable with λ in the abstraction. If there are n λs during the parsing between the binding λ and the occurrence of variable that place of variable is represented by n+1.
For eg. $(\lambda\ (\lambda y\ (\ldots\ (\lambda z\ \ldots))))$ is represented in de Bruijn notation as $(\lambda\ (\lambda\ (2\ (\lambda\ 1))))$.

If corresponding to a variable occurrence represented by n in a subterm M there is no nth closest surrounding λ then that variable occurrence is free in M. For eg. the underlined variable occurrences are free in $((\lambda\ (\underline{2}\ (\lambda\ (2\ \underline{3}))))\ (\lambda\ (\underline{4}\ 1)))$.

During the β-reduction of the redex (M N) the free variables of both the M and N are affected. Consider M = $(\lambda\ (\underline{2}\ (\lambda\ (\lambda\ 3))))$ and N = $(\lambda\ \underline{2})$ the underlined variable occurrences being free. Consider (M N) is the subterm of X = $(\lambda\ (M\ N))$. The free variables of M and N are referencing the outermost λ in X. When we reduce the redex (M N) in a naive fashion we get X = $\lambda\ (\underline{2}\ (\lambda\ (\lambda\ (\lambda\ \underline{2}))))$. This is wrong since now the free variables in M and N are not referencing the outermost λ in X. The free variables of M and N have to be properly modified. All the free variables of M have to be decremented by one since a λ is being destroyed in M. All the free variables of N have to be incremented by one as we traverse a λ recursively in M during the substitution since an extra λ is going to be added to the front of N. So the correct result is X = $\lambda\ (1\ (\lambda\ (\lambda\ (\lambda\ 4))))$.

Since variables do not have names there is no need for renaming bound variables during the sub titution. The de Bruijn

notation assumes importance in the computer implementation of λ-calculus and functional programming languages. One distinction with the standard notation is that if M is the subterm of X the order in which the free variable occurrences in M become bound in X is fixed regardless of X. So it is helpful to think of terms as always being subterms of some other bigger terms.

The λ-term with the de Bruijn notation can be formally defined as follows:

1.7 Definition (λ-terms with the de Bruijn notation) The set Λ of λ-terms is defined by

$\Lambda = \bigcup_{n \geq 0} \Lambda^n$

where Λ^n are the sets generated by the inductive rules

$k \in \Lambda^n$ if $1 \leq k \leq n$ (variables)

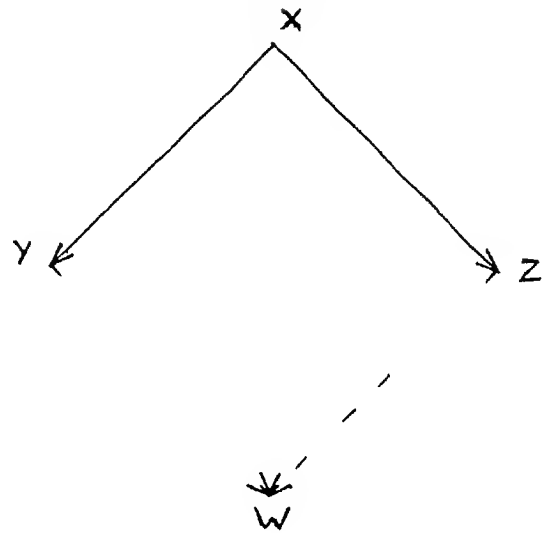
$\lambda N \in \Lambda^n$ if $N \in \Lambda^{n+1}$ (abstraction)

$(M N) \in \Lambda^n$ if $M, N \in \Lambda^n$ (application)

The closed terms are the terms in Λ^0. A detailed discussion of λ-calculus with this notation can be found in the appendix of [Bar80] and in [deB72].

2 Church-Rosser Theorem

For all X Y Z if $X \twoheadrightarrow Y$ and $X \twoheadrightarrow Z$
then there exists a W such that $Y \twoheadrightarrow W$ and $Z \twoheadrightarrow W$



The statement of the theorem can be shown as in the above figure and that is why it is also called diamond lemma The term X is called the source of the diamond Another way of stating this theorem is

All λ-terms can be the sources of diamonds

The Church-Rosser theorem assures the uniquness of the normal form i e all reduction paths do terminate at the same term if at all the they terminate

3 Walks relation

In this section we introduce another relation between two λ-terms called ***walks*** relation The proof of Church-Rosser consists in finding a simple relation which has diamond property such that reduction relation is the transitive closure of this new relation The first trial would be to consider β-step as this new relation since we already know that reduction is the transitive closure of

β-step But a single β-step relation does not have diamond property A counter example can illustrate this case Consider X=((λ)((λy y) a)) Y=((λ) a) and Z=(((λy y) a)((λy y) a)) We have that X reduces to Y and Z in a single β-step But now we can not find a W such that Y and Z reduces to it in a single β-step

Martin-Löf and Tait consider a relation consisting of a number of β- steps such that inner redexes are always contracted before the outer redexes They call it *special reduction* and [Sch88] calls it *walks* It is observed that this special reduction has the diamond property and that usual reduction is the transitive closure of this relation

3 1 Definition (Residual)

Let U be a term containing β-redexes P Q such that Q does not contain P Let U be the result of contracting Q in U If P Q are non-overlapping the residual of P in U is P itself If P=Q there will not be any residual of P in U If Q is a part of P and P=(λy M)N Then contraction of Q leaves (λy M)N or(λy M)N which is the residual of P in U

3 2 Definition (Special reduction or walk)

Consider R_1 R_n as redexes of X Contract any R_k which contains no other R_j This leaves n-1 residuals R_1 R_{k-1} R_{k+1} R_n Contract any R_j which contains no other R_k This process is continued till no residuals are left Then X is said to specially reduce or walk to the resulting Y

4 The main proof and lemmas involved

In this section we give an informal proof of Church-Rosser for standard notation of λ-calculus The proof for de Bruijn

notation needs few more lemmas regarding the incrementing of free variables and substitution A rigorous proof with this notation is given in the ne t chapter which is nothing but English translation of machine checked proof The proof involves three steps (i) Showing that walks follow diamond property (ii) Showing that the refle ive transitive closure of walks called n-walk follows diamond property (iii)Showing that diamond property of n-walks implie the diamond property of reduce relation

Step(i) The statement we need to prove is : *If X walks to Y and Z then there exists a W such that Y and Z walk to it* The proof is by induction on the structure of X

Case 1 (X is an atom) By definition Y and Z are also the same atom X Choosing W=X we have Y and Z walk to W

Case 2 (X=λ M) By definition there exist My M such that Y=λ My Z=λ M and M walks to My and M From the induction hypothesis for M we have Mw such that My and M walk to it Choosing the W=λ Mw will satisfy the requirements

Case 3 (X=(M N)) There arise four cases depending on whether X is a rede and if so whether it is the last rede to be β reduced in the walk to X and Y We consider here the case of X being a rede and in both the walks to Y and Z it is the last rede to be β-reduced The other three cases are dealt in the rigorous proof of ne t chapter

We have X=(λ A) Y=[Ny/]Ay Z=[N /]A such that A and N walk to Ay A and Ny N By the induction hypothese for A and N we have Aw Nw such that Ay A walk to Aw and Ny N walk to Nw Choosing [Nw/]Aw will satisfy our requirements given the following lemma

Lemma (Substitutivity of walks): If Ay walks to Aw and Ny walks to Nw then CNy/ ∃Ay walks to CNw/ ∃Aw The proof of the lemma is by induction on the structure of Ay and needs the following lemma

Lemma (sbst-sbst): CN/ ∃C /y∃M = C(CN/ ∃)/y∃(CN/ ∃M)

The proof of this lemma is by induction on the structure of the term M With the de Bruijn notation the proof of this lemma demands few more lemmas like incr-sbst incr-incr etc which are explained and proved in the next chapter The machine proof of the main theorem is contained in the Chapter walk diamond of machine proof

Step (ii) The proof that n-walks the reflexive and transitive closure of walks follows diamond property involves the induction on the length of walk chain This proof is shown in the machine proof as Chapter transitivity

Step (iii) The proof that diamond property of n-walks implies the diamond property of reduce relation consists of showing that a single walk can be considered as a series of β-steps and that a β-step can be considered as a single walk step The proof of these results are contained in Chapter c-r of machine proof

Chapter 3
Proof formalised in COC

In the previous chapter the proof of the Church-Rosser for standard notation is presented in an informal manner In this Chapter the formal and rigorous proof for Church-Rosser is pre ented The proof of this chapter is intended to be more or less an English translation of machin checked proof which is added a appendix B to thi thesi That is why the proof contains somewhat different notation from the standard literature For e ample a statement =D)/1Dy of standard literature is written as sbst(1 y) in the current Chapter which is actually represented in COC as (sbst 1 y) Throughout the chapter the variables which are not e plicitly bound in the statements of the lemmas and theorems are understood to be universally quantified

A constructive proof of Church-Ro ser formalised in Boyer-Moore logic is presented in [Sh88] We have followed this work closely to provide our proof in COC

The section 1 give the basic formulation of λ-calculus as has been done for the machine proof The section 2 gives some lemmas regarding the operations of incr_free_var and sbst Incr_free_var is the operation of incrementing the de Bruijn indices of the free variables of a term and sbst is the operation of substitution of a term in another term for some variable Most of the lemmas of this section are not needed for standard notation of λ calculus Hence these are absent in the standard literature The section 3 presents the lemma of substitutivity of walks The section 4 contains the proof that walk allows the diamond property The section 5 gives the proof that refle ive transitive closure of walks relation allows the diamond property The section 6 gives the proof of the Church-Rosser

1 Formulation of the basic theory of λ-calculus :

In this section the basic theory of λ-calculus which is needed for the proof of Church-Rosser is presented We define the λ-terms and subterms The operations of incrementing the free variables and substitution are explained Then we define the b- step and wall relations and their reflexive transitive closure All the definitions closely resemble their counter part in the machine proof For example the application of the predicate sbst term- nat- term term- prop to x y z and w is represented as sbst(y z w) while in the machine proof it is represented as (sbst y z w) This is done for the readability of the proof

1 1 Definition λ-terms :

A type *const* of constants is assumed to exist The type *nat* of natural numbers is defined inductively by the constructors zero and successor The type *term* of λ-terms is defined in the machine proof as

Inductive Definition term Data =

constant const->term

/variable nat->term

/lamda term->term

/apply term->term->term

For an element c of type const constant(c) is a term For a natural number v variable(v) is a term For terms t1 and t2 lamda(t1) and apply(t1 t2) are terms The symbol ***Pred*** stands for the predecessor function

From here onwards the exact synta of the definitions as in the machine proof is not given Only an English equivalent of the definitions is given The machine proof is presented in the appendix B We use the same names for predicates types functions and theorems as has been done in machine proof so that the

machine proof script is understandable

1 2 Definition Sub_term

The formula sub_term(x y) means that x is the sub_term of y The predicate *sub_term term->term->prop* is defined as follows

(i) For all terms sub_term(x x)

(ii) sub_term(x m) iff sub_term(x landa(m))

(iii)either sub_term(x m1) or sub_term(x m2) iff sub_term(x apply(m1 m2))

1 3 Definition (Incr_free_var) :

With de Bruijn notation in the β-reduction of $(\lambda X)Y$ the free variables of Y are affected For example we have $X=\lambda 2$ and $Y=1$ The term Y has a free variable which is bound by the nearest surrounding λ If we substitute Y in X for the first bound variable in a naive fashion the β-reduction gives us $\lambda 1$ But this is wrong since substitution resulted in a term which does not have any more free variables The free variable of Y got bounded in X during the substitution The corect result of substitution is $\lambda 2$ So during the substitution the free variables of Y have to be incremented by one each time we recurs down a λ in the structure of X

The operation of incrementing the free variables of Y is defined by induction on the structure of Y This operation can be defined as a two place function with the arguments the term and a counter to keep track of the number of λs surrounding the subterm under consideration The formula in r_free_var(i y) means that the free variables of x are the variables which are bound beyond i levels and these variables are incremented by one to get the term y The initial value of the counter has to be zero

The predicate *incr_free_var* of the type term- nat- term- Prop is defined recursively on the structure of

the term whose free variables are being incremented is follows

(a) If x = constant(c)
then incr_free_var(x y) iff y = constant(c)

(b) If x = variable(v) and x v
then incr_free_var(x y) iff y = variable(S(v))

(c) If x = variable(v) and i≥v
then incr_free_var(x y) iff y = variable(v)

(d) If x = lambda(a)
then incr_free_var(x y) iff
y = lambda(n1) and incr_free_var(a S(x) a1)

(e) If x = apply(m1 n1)
then incr_free_var(x x y) iff y = apply(m2 n2)
incr_free_var(m1 x m2) incr_free_var(n1 x n2)

1 4 Definition Sbst :

During the β-reduction of (λ X)Y the free variables of X are also affected since the λ in X is lost in course of β-reduction This would mean that there is one fewer λ surrounding the free variables of X For example consider X=λ (3 (2 1)) Here 1 is the bound variable X 2 is the variable for replacement during the β-reduction and 3 is the free variable If we reduce in a naive fashion we get λ (3 (Y 1)) But the correct result is λ (2 (Y 1)) The free variable 3 has to be decremented by one since a surrounding λ is lost During the recursive definition of substitution a counter is maintained to recognise the free variable occurrences

The formula sbst(x y w) means that y is the variable to be replaced all the variables less than y are bound in which are to be kept as they are all the variables greater than y are free which are to be decremented and w is the result of substituting for the occurrence of y in The predicate sbst of type term nat term- term- prop is defined recursively on the

structure of the term in which the substitution is being performed as follows :

(a) If $-constant(c)$ then
$sbst(y\ w)$ iff $w = constant(c)$

(b) If $-variable(v)$ and $y = v$ then
$sbst(y\ w)$ iff $w = variable(Pred(v))$

(c) If $-variable(v)$ and $v \neq y$ then
$sbst(y\ w)$ iff $w = variable(v)$

(d) If $= variable(v)$ and $y = v$ then
$sbst(y\ w)$ iff $w = y$

(e) If $= lambda(a)$ then
$sbst(y\ w)$ iff $w = lambda(a')$ $incr\_free\_var(0\ y)$ and $sbst(a\ z(y)\ a\ a')$

(f) If $= apply(m1\ n1)$ then
$sbst(y\ w)$ iff $w = apply(m2\ n2)$ $sbst(y\ m1\ m2)$ and $sbst(y\ n1\ n2)$

1.5 Definition b_step :

The formula $b\_step(y)$ means that y is got by β-reducing x in a single step The predicate b_step of the type term- term- prop is defined recursively on the structure of the term which is being β-reduced as follows

(a) If $= lambda(m)$ then
$b\_step(y)$ iff $y = lambda(m')$ and $b\_step(m\ m')$

(b) If $= apply(m\ n)$ then
$b\_step(y)$ iff either

(i) $y = apply(m1\ n)$ and $b\_step(m\ m1)$

or

(ii) $y = apply(m\ n1)$ and $b\_step(n\ n1)$

or

(iii) $m = lamda(a)$ and $sbst(n\ one\ a\ y)$

1 6 Definition n_b_step

The relation n_b_step is the reflexive and transitive closure of relation b_step. The formula n_b_step(y n) means that b-reduces to y in n b-steps. The predicate n_b_step of the type term- term- nat prop is recursively defined on the length of the b_step chain a follows

(a) Reflexivity over a single b_step

n_b_step (u v 0) iff u = v

(b) Transitivity over a single b_step

n_b_step (u w S(n)) iff n_b_step (u v n) and b_step(v w)

1 7 Definition reduce :

A term x is said to reduce to y iff there exists a number n such that n_b_step(y n) holds

1 8 Definition Walks :

The motivation for defining a special reduction walk has been explained in section 3 of Chapter 2. The proof of Church-Rosser consists in showing that the reflexive transitive closure of walk relation allows diamond property and in showing that a b-step can be seen as a walk and a walk can be seen as a series of b-steps. The formula walks(y) means that walks to y or in other words transforms to y in a finite number of b reductions performed in an inside first manner.

The relation walks of type term- term prop is defined recursively on the structure of the term which is being walked from as follows

(a) If = constant(c) then

walks(y) iff y = constant(c)

(b) If x = variable (v) then

walks(x y) iff y = variable(v)

(c) If = lambda(m) then
walks(x y) iff y = lambda(m1) and walks(m m1)
(d) If = apply(m1 n1) then
wall (y) iff either
(i) y = apply(m2 n2) walks(m1 m2) and walks(n1 n2)
or
(ii) m = lambda(a1) m2 = lambda(n2) walks(m1 m2)
walks(n1 n2) and sbst(n2 one a2 y)

1 9 Definition n_walks :

n_walks is the reflexive transitive closure of walks relation The formula n_walks(y n) means that walks to y in n wall teps The relation n_walks is defined recursively on the length of the walk chain as follows

(a) Reflexivity over walks relation
n_walks (u v 0) iff u v
(b) Transitivity over walks relation
n_walks (u w S(n)) iff n_walls (u v n) and walls (v w)

1 10 Induction schemas

We have following two schemas of induction on the structure of term

Axiom sub_term_induction (∀x term)(∀P:term Prop)
((∀c const) P(constant(c)))-
((∀v nat) P(variable(v)))-
((∀m term)((:term) sub_term(m)- P(x))-
P(lambda(m)))-
((∀m1 term)(∀m2 term)
((∀ term) sub_term(m1)- P( ))
((∀y:term) sub_term(y m2)- P(y))-
P(apply(m1 m2)))-
P( )

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Axiom trm_induction : (∀ :term)(∀P term  Prop)
                      ((∀c con t) P(constant(c)))
                      ((∀v nat) P(variable(v)))-
                      ((∀m t rm) P(m)-  P(lambda(m)))
                      ((∀m1 term)(m2 term) P(m1)-  P(m2)-
                                           P(apply(m1 m2)))-
                      P( :)
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The schema sub_term_induction is the stronger of the two in the sense that term_induction can be derived from it and not vice versa

2 Some properties of incr_free_var and sbst :

This section presents some properties of incr_free_var and sbst In all the lemmas the cases which are stated to be *trivial* actually result in absurdities Anything can be proved from absurdity Hence the proofs are trivial

(a) Lemma sbst_not_free_in :

This lemma says that if we increment the free variables of a term bound beyond i levels then the term is free for the variable of S(i) th level The statement of the lemma is :

incr_free_var(y i y)->sbst(x (S i) y w)->(w=y)

Proof : The proof is by induction on the term y

We have incr_free_var(y i y) (1)

sb t(S(i) y w) (2)

We have to prove w=y

Variable case: We have y=variable(v) (3)

For v i _ The variable(v) in the eq (1) is bound so from the

definition of incr_free_var y =variable(v) This variable is once again bound in the eq (2) so w=variable(v)

For v= 1 : The variable(v) is bound in the eq (1) so y =variable(v) Once again this variable is bound in the eq (2) hence w=variable(v)

For v 1 _ The variable(v) is free in the eq (1) so y =variable(v+1) Once again this variable is free in the eq (2) so from the definition of bst w=variable(v)

Lambda case_ We have y=lambda(m) (4)
From the definition of incr_free_var for lamda case and eq (1) there e ists m such that

y =lamda(m) incr_free_var(m S(1) m) (5)

From the definition of sbst and eq (2) there e ist < and m such that

incr_free_var(< 0) w=lamda(m) sbst(S(S(1)) m m) (6)

From the induction hypothesis and eq (5)&(6) we have m=m Hence w=lamda(n)

Apply case: We have y=apply(m1 m2) We have the appropriate induction hypotleses for the terms m1 and m2 From the eq (1) we have

y =apply(m1 m2) incr_free_var(m1 i m1)
incr_free_var(m2 i m2) (7)

From the eq (2)&(7)

w=apply(m1 m2) sbst(< S(i) m1 m1)
sbst(S(i) m2 m2) (8)

From the inductive hypotheses we have m1=m1 and m2=m2 Hence w=apply(m1 m2)

(b)Incr_incr lemma

The lemma shows how the order of two incr_free_var can be e changed The statement of th1 lemma is

$$\langle i\ j\ nat\rangle\ le(j\ i)\rightarrow$$
$$incr\_free\_var(x\ i\ xi\ )\rightarrow incr\_free\_var(xi\ \ j\ \ xi\ \ )\rightarrow$$
$$ncr\_free\_var(x\ j\ x2\ )\rightarrow\ ncr\_free\_var(x2\ \ S(i)\ \ xi\ \ )$$

Proof The lemma is proved by induction on the tructure of the term We have

$$le(j\ i) \quad (1)$$
$$incr\_free\_var(\ \ i\ \ \ 1\ ) \quad (2)$$
$$incr\ free\_var(\ 1\ \ j\ \ \ 1\ \ ) \quad (3)$$
$$incr\_free\_var(\langle\ j\ \ \ 2\ ) \quad (4)$$

We have to show incr_free_var(<2 S(1) 1)

Variable case Different cases arise depending on the diff rent values of i and j

For i v, j v : The variable v is free in eq (2) So 1 =variable(S(v)) This variable is again free in the eq (3) So 1 =variable(v+2) The variable v in the eq (4) is free and so x2 =variable(S(v)) From the definition of incr_free_var we have that incr_free-var(2 S(1) 1) holds

For i v, j=v _ This case is trivial

For i v, j v _ This case is trivial

For i=v, j v _ In the eq (2) the variable v is bound and so 1 =variable(v) This variable is free in the eq (3) So 1 =variable(S(v)) In the eq (4) the variable v is free and so <2 =variable(S(v)) From the definition of incr_free_var we have that incr_free_var(x2 S(1) <1) holds

For i=v, j=v _ In all the eq (2) (4) the variables of substitution are bound So 1 =variable(v) 1 variable(v) 2 =variable(v) Henc incr_free_var(2 S(1) <1) holds

For i v, j v : This case is trivial

For i v, j v _ The variable v in eq (2) is bound so 1 =variable(v) This variable in the eq (3) is free so 1 =variable(S(v)) In the eq (4) the variable v is free so 2 =variable(S(v)) From the definition of incr_free_var it holds that incr_free_var(2 S(1) <1)

For i v, j=v _ In all the eq (2)-(4) the variable is bound so we have <1 =variable(v) 1 =variable(v) and <2 =variable(v) Hence incr_free_var(<2 S(1) 1)

For i v, j v _ This case is similar to the previous case

Lambda case Here we have lam la(m) From the inductive hypothesis for m the proof is straight forward

Apply case : The proof follows from the inductive hypotheses

(c)Incr_sbst_lemma1 :

This lemma shows how the incr_free_var distributes over the sbst Without the restriction lt(i j) the lemma can not be proved

lt(i j)->sbst(x j y w1)->incr_free_var(w1 i w2)->
incr_free_var(x x)->incr_free_var(y i y)->
sbst(x S(j) y w2)

Proof : This lemma is proved by induction on the structure of the

term y We have

$$lt(i\ j) \quad (1)$$

$$bst(x\ j\ y\ w1) \quad (2)$$

$$incr\_free\_var(w1\ i\ w2) \quad (3)$$

$$incr\_free\_var(\ i\ ) \quad (4)$$

$$incr\_free\_var(y\ i\ y\ ) \quad (5)$$

We have to prove that sbst(S(j) y w2)

Variable case : We have y=variable(v) The various cases for the different values of i j are shown below

For j v, S(i) v _ The variable v in eq (2) i free so w1=Pred(v) This variable in eq (3) is again free so w2=variable(v) The variable v in eq (5) is also free o y =S(v) Hence we have sbst() S(j) y w2)

For j v, S(i)=v _ This case is trivial

For j v, S(i) v : This case is also trivial

For j=v, S(i)<v : The variable v is the actual variable for substitution in eq (2) so w1=x So from the eq (3)&(4) w2=) The variable in eq (5) is free so y =variable(S(v)) Hence sbst(x S(j) y w2)

For j=v, S(i) v : The situation is similar to the previous case

For j=v, S(i) v _ This case is trivial

For j v, i v : The variable v is bound in eq (2) so w1=variable(v) This variable is free in eq (3) so w2=variable(S(v)) The variable v is free in eq (5) so

y =variable(S(v)) Hence sbst(x S(j) y w2)

For j v i=v : The variable v is bound in the eq (2) so w1-variable(v) This variable is again bound in eq (3) so w2-variable(v) The variable v is bound in eq (5) so y -variable(v) Hence sbst(x S(j) y w2)

For j v, i v _ The situation is similar to the previous case

Lambda case : Here we have y=lamda(m) From eq (2) there exist x0 m0 such that

w1-lamda(m0) incr_free_var(x 0 x0) sbst(x0 S(j) m m0) (6)

From the eq (3)&(6) there exists m1 such that

w2=lamda(m1) and incr_free_var(m0 S(i) m1) (7)

From the eq (5) there exists m2 such that

y =lamda(m2) and incr_free_var(m S(i) m2) (8)

We can find such that

incr_free_var(0) (9)

From incr_incr_lemma and eqs (4) (6)&(9) we conclude incr free_var(x0 S(i)) From the inductive hypothesis for m sbst(S(S(j)) m2 m1) Hence from the eqs (7) (8)&(9) we can conclude that sbst(x S(j) y w2)

Apply case : We have y=apply(m1 m2) From the eq (2)-(5) there exist m1 m1 m1 m2 m2 m2 such that

w1=apply(m1 m2) sbst(x j m1 m1) sbst(j m2 m2) (10)

w2=apply(m1 m2) incr_free_var(m1 i m1)

incr_free_var(m2 i m2) (11)

y =apply(m1 m2)incr_free_var(m1 i m1)

incr_free_var(m2 i m2) (12)

From the inductive hypotheses for m1 and m2 we can infer sbst(x S(j) m1 m1) and sbst(y S(j) m2 m2) Hence from

the definition of sbst sbst(x S(j) y w2)

(d) Incr_sbst_lemma2 :

This is again another distribution of incr_free_var over sbst The minor variation from incr_sbst_lemma1 has to be noticed carefully

le(i S(j))->sbst(m i a w1)->incr_free_var(w1 j w2)->
ncr_free_var(a S(j) a)->incr_free_var(m j m)->
sbst(m i a w3)-> (w2=w3)

Proof : The proof is by induction on the structure of the term a We have le(i S(j)) (1)

slst(m i a w1) (2)

incr_free_var(w1 j w2) (3)

incr_free_var(a S(j) a) (4)

incr_free_var(m j m) (5)

bst(m i ? w3) (6)

Variable case _ We have a-variable(v) The various cases arising depending on various values i and j are shown below :

For i v, S(j) v _ The variable v is free in the eq (2) so w1=variable(v-1) This variable is again free in eq (3) so w2=variable(v) The variable v in eq (4) is also free so a =variable(v+1) This variable is again free in eq (6) so w3=variable(v) So w2=w3

For i v, S(j)=v _ The variable v is free in the eq (2) so w1=variable(v-1) This variable is bound in eq (3) so w2 variable(v-1) The variable v is bound in eq (4) so a =variable(v) This variable is free in eq (6) so w3-variable(v-1) So w2=w3

For i v, S(j) v : The situation is very similar to the previous case

For i=v, S(j) v : This case is trivial

For i=v, S(j)=v : The variable v in eq (2) is the actual variable to be replaced so w1=m and w2=m The variable v is bound in eq (4) so a =variable(v) This variable in eq (6) is the actual variable for substitution So w3=m Hence w2=w3

For i=v, S(j) v : This case is similar to the previous one

For i v, S(j) v : This case is trivial

For i v, S(j)=v : This case is also trivial

For i v, S(j) v : The variable v in eq (2) is bound so w1=variable(v) The j can be greater than or equal to v So the variable in the eq (3) is bound so w2=variable(v) The variable v is bound in eq (4) so a =variable(v) This variable is again bound in eq (6) so w3=variable(v) Hence w2=w3

Lambda case : We have a=lamda(m) From eq (2)-(6) there exist m0 m1 m1 m2 m2 such that

w1=lamda(m1) incr_free_var(m 0 m0) sbst(m0 S(i) m m1) (7)

w2=lamda(m1) incr_free_var(m1 S(j) m1) (8)

a =lamda(m2) incr_free_var(m S(S(j)) m2) (9)

w3=lamda(m2) incr_free_var(m 0 m) sbst(m S(i) m2 m2) (10)

From incr_incr_lemma and eq (5) (7) (10) we have

incr_free_var(m0 S(j) m) (11)

From the inductive hypothesis for n we conclude that m1 m2 Hence w2-w1

Apply case _ Here we have a=apply(m1 m2) From the inductive hypotheses for m1 and n2 the proof of this case is straight forward

(*) Lemma Incr_walk :

incr_free_var(n1 i n1) -> ncr_free_var(n2 i n2) ->

walks(n1 n2) -> walks(n1 n2)

This lemma tells us how the walks relation behaves with respect to incr_free_var This lemma is needed in the proof of lambda case of sbst_lemma

Proof : The lemma is proved by induction on the structure of the term n1 We have

incr_free_var(n1 i n1) (1)

incr_free_var(n2 i n2) (2)

walks(n1 n2) (3)

Variable case: Here we have n1=variable(v) From eq (3)

n2=variable(v) (4)

Three cases arise depending on the relation between i and v

For i v _ The variable v is free in the eq (1)&(2) So n1 =variable(S(v)) and n2 =variable(S(v)) From the definition of walks n1 walks to n2

For i=v _ The variable v is bound in the eq (1)&(2) So n1 =variable(v) and n2 =variable(v) From the definition of walks n1 walks to n2

For i v _ the case i very similar to the previous one

Lambda case Here n1=lambda(m) from the inductive hypothesis for term n and the definition of walk the proof is obvious

Apply case Here we have n1=apply(m1 m2) From the eq (1) there exist m1 m2 such that

$$n1 = apply(m1\ m2)\ \ incr\_free\_var(m1\ i\ m1)\ \ incr\_free\_var(m2\ i\ m2) \quad (5)$$

from the definition of walks two subcases arise :

Subcase (a) $n2 = apply(m3\ m4)\ \ walks(m1\ m3)\ \ walks(m2\ m4)$ (6)
From the eq (2)&(6) there exist m3 m4 such that

$$n2 = apply(m3\ m4)\ \ incr\_free\_var(m3\ i\ m3)\ \ incr\_free\_var(m4\ i\ m4) \quad (7)$$

From inductive hypotheses for m1 m2 and eq (6)&(7) walks(m1 m3) and walks(m2 m4) Hence n1 walks to n2

Subcase (b) There exist a a1 m3 m4 such that

$$m1=lambda(a)\ \ m3=lambda(a1)\ \ walks(m1\ m3)\ \ walks(m2\ m4)\ \ subst(m4\ one\ a1\ n2) \quad (8)$$

We can find terms m3 m4 n2 a1 such that

$$incr\_free\_var(m3\ i\ m3) \quad (9)$$

$$incr\_free\_var(m4\ i\ m4) \quad (10)$$

$$m3 = lambda(a1)\ \ incr\_free\_var(a1\ S(i)\ a1) \quad (11)$$

$$subst(m4\ one\ a1\ n2) \quad (12)$$

From the lemma incr_subst_lemma2

$$n2 = n2 \quad (13)$$

From the inductive hypothesis for m1 m2 we infer that

$$walks(m1\ m3)\ walks(m2\ m4) \quad (14)$$

From the definition of walk and eq (12)&(14) walks(n1 n2) Hence from eq (13) n1 walks to n2

(f) Sbst_sbst_lemma

This lemma shows how two successive sbst operations can be exchanged. The restriction that $j \leq i$ is warranted in the proof of the lemma.

$$le(j\ i) \rightarrow sbst(m2\ j\ m1\ w1) \rightarrow sbst(n2\ i\ w1\ w2) \rightarrow$$
$$ncr\_free\_var(n2\ Pred(j)\ n2) \rightarrow sbst(n2\ S(i)\ m1\ m1) \rightarrow$$
$$sbst(n2\ i\ m2\ m2) \rightarrow sbst(m2\ j\ m1\ w3) \rightarrow (w2=w3)$$

When written in usual notation for $j \leq i$

$$Ln2/i](Cn2/j]m1 = L\ Cn2/i]m2)/j](Cn2/S(i)]m1)$$

Proof : This lemma is proved by induction on the structure of the term m1.

le(j i)	(1)
sbst(m2 j m1 w1)	(2)
sbst(n2 i w1 w2)	(3)
incr_free_var(n2 j 1 n2)	(4)
sbst(n2 (S i) m1 m1)	(5)
sbst(n2 i m2 m2)	(6)
sbst(m2 j m1 w3)	(7)

Variable case: For different values of i j the value of w2 and w3 are shown to be equal.

For j v, i v, S(i) v : In the eq (2) variable(v) is free so w1=variable(v-1). This variable is free in eq (3) so w3=variable(v-2). In eq (5) the variable(v) is free so we have m1 =variable(v-1). When S(j) v the variable(v-1) in the eq (7) is free so w3=variable(v-2). So w2=w3. The case S(j)≥v is not possible.

For j v, i v, S(i)=v : As in the previous case w1=variable(v-1)

In the eq (3) variable(v-1) is the variable for substitution So w2=n2 In eq (5) the variable(v) is the variable for substitution so m1 =n2 So from the lemma sbst_not_free and eq (4)&(7) we have w3=n2 Hence w2=w3

For j v, i < v, S(i) v _ This case is trivial

For j v, i=v _ In the eq (2) the variable(v) is free so w1=variable(v-1) In the eq (3) this variable is bound o w2=variable(v-1) In eq (5) the variable(v) is bound so m1 =variable(v) This variable is free in eq (7) so w3=variable(v-1) Hence w2=w3

For j v, i v : The situation is very similar to the previous case and we have w2=w3 variable(v-1)

For j<v, i < v _ This case is trivial

For j=v, i=v : In eq(2) the variable(v) is the actual variable to be replaced so w1=m2 So w2 = [n2/i]m2 In the eq (5) the variable(v) is bound so m1 =variable(v) This variable is the actual variable to be replaced in the eq (7) and so w3 = m2 i e w3 = [n2/i]m2 Hence w2=w3

For j=v, i v _ The situation is very much same to the previous case We have w2=w3=[n2/i]m2

For j v, i v _ This case is trivial

For j v, i=v : This case is also trivial

For j v, i v _ In all the eq (2) (3) (5)&(7) the variable v

is found Hence w2=w3=variable(v)

Lambda case Here we have m1=lambda(m) The proof of this lemma exemplifies the utility of a proof checker in complex proving We have the induction hypothesis that

(m2 m1 m2 w1 w2 w3 n2 n2 term)(i j:nat)
le(j i)- subst(m2 j m w1)- subst(n2 i w1 w2)-
incr_free var(n2 j-1 n2)- subst(n2 S(i) m m1)-
subst(n2 i m2 m2)- subst(m2 j m1 w3)- w3 w2 (8)

From the definition of sub t and eq (2) there must exist m21 w11 such that

incr_free var(n2 0 m21) (9)
w1=lambda(w11) (10)
subst(m21 (j) m w11) (11)

From the definition of subst and eq (3)&(10) there exist n21 w21 such that

incr_free_var(n2 0 n21) (12)
w2=lamda(w21) (13)
subst(n21 S(i) w11 w21) (14)

From the definition of subst and eq (5) there exist n2 m12 such that

incr_free_var(n2 0 n2) (15)
m1 =lambda(m12) (16)
subst(n2 S(S(i)) m m12) (17)

From the definition of subst and eq (7)&(16) there exist m22 w32 such that

incr_free_var(m2 0 m22) (18)
w3=lambda(w32) (19)
subst(m22 S(j) m12 w32) (20)

From the incr_incr_lemma and eq (4) (12)&(15) we have that

incr_free_var(n21 j n2) (21)

From the incr_subst_lemma1 and eq (6) (9) (12)&(18) we have that

sbst(n21 S(1) π21 m22) (22)

From the induction hypothesis (8) and eq (11) (14) (17) (20) (21) (27)

w21=w32 (23)

From the eq (13) (19)&(23) we have w2=w3

Apply case Here we have π1=apply(m3 π4) We also have the appropriate induction hypothesis for the term π3 π4 From the definition of sbst and eq (2) there exist m3a m4a such that

w1=apply(π3a m4a) (24)

sbst(π2 3 π3 m3a) (25)

sbst(π2 3 π4 m4a) (26)

From the definition of sbst and eq (3)&(24) there exist π3b m4b such that

w2=apply(π3b m4b) (27)

sbst(n2 1 m3a m3b) (28)

sbst(n2 1 π4a π4b) (29)

From the definition of sbst and eq (5) there exist m3c π4c such that

π1 =apply(π3c π4c) (30)

sbst(n2 S(1) π3 m3c) (31)

sbst(n2 S(1) π4 m4c) (32)

From the definition of sbst and eq (7) there exist m3d π4d such that

w3=apply(m3d π4d) (33)

sbst(m2 3 m3c π3d) (34)

sbst(π2 3 π4c π4d) (35)

From the induction hypothesis for m3 π4 and eq (24)-(35) we have that π1b=π1d and m2b=m2d Hence w2=w3

3 Sbst_lemma : (Substitutivity of walks)

This lemma asserts that if a1 and n1 walk to a2 and n2

respe tively then the result of substituting n1 in a1 for i walks to the result of sub tituting n2 in a2 for the same variable

walks(a1 a2) -> walks(n1 n2) -> sbst(n1 i a1 w1) ->

sbst(n2 i a2 w2) -> walks(w1 w2)

This lemma is crucial for the proof that the walks relation follows diamond property

Proof

This lemma is proved by induction on the structure of th term a1 Here we have

walks(a1 a2) (1)

walks(n1 n2) (2)

sbst(n1 i a1 w1) (3)

sbst(n2 i a2 w2) (4)

Variable ca e Here we have a1=variable(v) From the definition of walks and eq (1) we have a2 = variable(v) Here three cases arise depending on the relation between i and v

For i v : In the eq (3)&(4) the variable v is free so from the definition of sbst we have w1 = Pred(v) w2 = Pred(v) So from the definition of walks w1 walks to w2

For i=v _ In both the eq (3) and (4) the variable v is the actual variable to be substituted So from the definition of walks w1=n1 and w2=n2 From eq (2) w1 walks to w2

For i v _ In both the eq (3) and (4) the variable v is bound So from the definition of sbst w1=variable(v) and w2=variable(v) Hence w1 walks to w2

Lambda case Here we have a1=lamda(m) From the definition of walks

and eq (1) there exist m1 such that

a2=lamda(π1) and walks(π n1) (5)

From the definition of sbst and eq (3) (4)&(5)

incr_free_var(n1 0 n1) w1=lamda(w1) sbst(n1 S(1) π w1) (6)

incr_free_var(n2 0 n2) w2=lamda(w2) sbst(n2 S(1) m w2) (7)

From the lemma incr walk and eq (6)&(7)

walks(n1 n2) (8)

From the inductive hypothesis for m and eq (5) (7) we infer walks(w1 w2) Hence from eqs (6)&(7) walks(w1 w2)

Apply case: Here we have a1=apply(m1 π2)

Here two cases arise from the definition of walk

Subcase (a): There exist π3 and m4 such that

a2=apply(π3 π4) walks(π1 π3) walks(m2 m4) (9)

From the definition of sbst eq (3)&(4) there exist π1 π2 π3 π4 such that

w1 apply(π1 π2) sbst(n1 i π1 π1) bst(n2 1 π2 π2) (10)

w2=apply(m3 π4) st t(n1 i m3 π3) sbst(n2 1 π4 π4) (11)

From the inductive hypotheses for m1 and π2 we infer that m1 walks to m3 and π2 walks to π4 Now from the definition of walks w1 walks to w2

Subcase (b) There exist terms a a1 π3 m4 such that

π1=lamda(a) π3=lamda(a1) walks(a a1) walks(π1 π3)

walk (π2 π4) sbst(π4 one a1 a2) (12)

From the eq (3) and definition of sbst there exist π1 π2 such that

w1=apply(π1 π2) sbst(n1 1 π1 π1) sb t(n2 1 π2 m2) (13)

We can find terms n2 π4 a1 w such that

incr_free_var(n2 0 n2) (14)

sbst(n2 1 π4 π4) (15)

subst(n2 S(x) a1 a1) (16)

subst(m4 one a1 w) (17)

From the eq (13) (17) and lemma sb t sb t_lemma we can infer that w =w2 From the inductive hypotheses for m1 m2 we have

walks(m1 lamda(a1)) and walks(m2 n4) (18)

From the definition of walks and eq (17)&(18)

walks(apply(m1 m2) w) (19)

Since w =w2 from the eq (19) we have walks(w1 w2)

4 Walks follows diamond property (Theorem diamond_walks)

In this section we prove that the relation walks follows diamond property The theorem diamond_walks reads as

(x term) diamond (x)

It states that all lambda calculus terms can be the sources of diamonds Another form of the statement is

walks(x y) and walks(x z) =>

∃(w term)(walks(y w) ∧ walks(z w))

Proof: The proof is by induction on the structure of the term (i e the term which is being walked from We make use of sub_term_induction instead of term_induction The sub_term_induction is a stronger version of induction on the structure of the term The use of sub_term_induction is actually necessitated in the case when =apply(m n) m=lamda(m) for some m n m and the last redex β_reduced in the walk is apply(m n) In this case if we had made use of term_induction we have the induction hypothesis only for the term m But we need a stronger induction hypothesis holding for the sub_term m of m It is sure that we can prove that the term m can become the source of a diamond if we have that m is the source of a diamond and m=lamda(m) once we have walk_diamond But we are actually proving the diamond property itself and hence can not deduce that m can

be the source of a diamond if n 1 one such So we need the stronger version of induction

We have walks(y) and walks( ) (1)

Variable case : We have = variable(v) If walk to y and from the definition of walks we have that

y = variable(v) and = variable(v) (2)

Choose w as variable(v) From the definition of walks and from the eq(2) we have that both y and walk to w

Lambda case : We have = lambda(m) If walks to y and from the definition of walks there must exist terms m1 and m2 such that

y = lambda(m1) and walks(m m1) (3)

= lambda(m2) and walks(m m2) (4)

Induction hypothesis says that all the sub_terms of m and n can be the sources of diamonds This means that m and n themselves can be the sources of the diamonds From this and eq (3) and eq (4) it follows that there must exist a term m3 such that

walks(m1 m3) and walks(m2 m3) (5)

Choose w = lambda(m3) From the definition of walk and eq (3) (4)&(5) we have that both y and z walk to w

Apply case : Here we have =apply(m n) We have walks(y) and walks(x) From the definition of walks there arise four subcases depending on whether or not m is an abstraction and if it is whether or not the redex apply(m n) is the last redex to be β-reduced in the walk

Subcase (a) Apply(m n) is not the last redex to be β-reduced in both the walks to y and From the definition of walks if x walks to y and then there must exist terms m1 n1 m2 n2 such that

y = apply(m1 n1) and = apply(m2 n2) (6)

walks(m m1) and walks(n n1) (7)

walks(m m2) and walks(n n2) (8)

We have the induction hypotheses that all subterms of m and n can be the sources of diamonds This obviously means that the terms m and n can themselves be the sources of diamonds From this conclusion and eq (7)&(8) there must exist w1 and w2 such that

walks(m1 w1) and walks(m2 w1) (9)

walks(n1 w2) and walks(n2 w2) (10)

Now choose w = apply(w1 w2) From eq (6) (9)& (10) and from the definition of walks we can conclude that y of the form apply(m1 n1) and of the form apply(m2 n2) walks to w of the form apply(w1 w2)

Subcase (b) Apply(m n) is the last redex to be β-reduced in the walk to x while it is not in the walk to y From the definition of walks if walks to y then there must exist terms m1 and n1 such that

y = apply(m1 n1) walks(m m1) and walk (n n1) (11)

From the definition of walks and the eq (11) there must exist a term a1 such that

m1 = lamda(a1) and walks(a a1) (12)

From the definition of walk if walks to then there must exist terms a a2 m2 n2 such that

m =lamda(a) m2=lamda(a2) walks(a a2) walks(m m2)

walk (n n2) and sbst(n2 one a2) (13)

From the induction hypotheses that all the sub-terms of m and n can be the sources of the diamonds the terms a n can be the sources of diamonds From this and the eq (11) (12)&(13) there exist terms a3 n3 such that

walks(a1 a3) and walks(a2 a3) (14)

walks(n1 n3) and walks(n2 n3) (15)

Now choose a term w such that sbst(n3 one a3 w) From the definition of walks and the eq (11) (12) (14)&(15) we conclude that the term y of the form apply(π1 n1) walks to the term w of the form [n3/1]a3 From the sbst_lemma (substitutivity of walks) and eq (13) (14) (15) we conclude that the term of the form [n2/1]a walk to the term w of the form [n3/1]a3

Subcase (c) Apply(π n) is the last redex to be β-reduced in the walk to y while it is not in the walk to The proof is similar to the last case

Subcase (i) _ Apply(π n) is the last redex to be β-reduced in both the walks to y and From the definition of walks there must exist terms a a1 π1 n1 a0 a2 m2 n2 such that

π = lamda(a) π1 = lamda(a1) walks(a a1)

walks(π π1) walks(n n1) sbst(n1 one a1 y) (16)

π lamda(a0) π2 = lamda(a2) walks(a0 a2)

walks(π π2) walks(n n2) and sbst(n2 one a2) (17)

Since π=lamda(a)=lamda(a0) we have a=a0 Hence walks(a a2) (18)

The induction hypotheses say that all the sub-terms of the terms m and n can be the sources of diamonds The terms a and n can be sources of diamonds From this and eq (16) eq (17) eq (18) we can conclude that there exist terms a3 m3 such that

walks(a1 a3) and walks(a2 a3) (19)

walks(n1 n3) and walks(n2 n3) (20)

Now choose a term w such that sbst(n3 one a3 w) From the sbst_lemma (substitutivity of walks) eq (16) eq (17) eq (18) eq (19) eq (20) we can infer that y of the form [n1/1]a1 and the form [n2/1]a2 walks to w of the form [n3/1]a3

5 Reflexive Transitive Closure of Walks & Diamond Property :

The relation n_walks is the reflexive and transitive closure

of walks relation The main result of the section is that n_walks also follows diamond property Before proving this we prove a small lemma

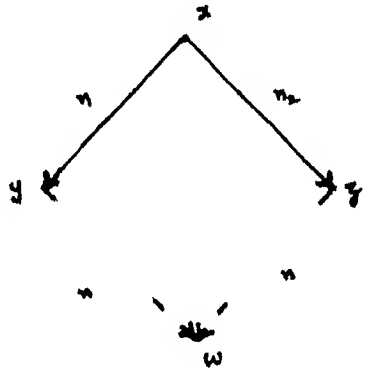
5 1 Lemma m_one_diamond_one_m :

$$n\_walks(x\ y\ n) \rightarrow walks(x\ z) \rightarrow \exists(w\ term)\ (walks(y\ w) \wedge n\_walks(z\ w\ n))$$

This lemma is a special case of the subsequent theorem in the sense that the length of second walk chain is fixed at one Hence the proof is not given

5 2 Theorem trans_diamond :

$$n\_walks(x\ y\ n1) \rightarrow n\_walks(x\ z\ n2) \rightarrow \exists(w\ term)\ n\_walks(y\ w\ n2) \wedge n\_walks(z\ w\ n1)$$



The statement of this theorem can be shown by the diagram above

Proof : The theorem is proved by a double induction on the lengths of walk chains from x to y and i e n1 and n2 We have

n_walks(x y n1) and n_walks(x z n2) (1)

Case 1 (ero_case) The lengths of walk chains are ero If x walks to y and z in 0 steps from the definition of n_walks for reflexive case

y = and = (2)

Now choose w= Now from the definition of n_walks and eq (2) we know that n_walks(y w O) and n_walks(w O)

Case _ _ (ind1_case) We have the induction hypothesis that for one fixed n1 n2

(y term) n_walks(y n1) - n_walks(n2) -

∃(w term) (n_walks(y w n2) ^ n_walks(w n1)) (3)

We are given that for a particular <

n_walks(< y S(n1)) and n_walk (< z n2) (4)

Now we have to exhibit a w such that y and walk to it in n2 and S(n1) walk steps respectively From eq (4) and the definition of n_walks for transitive case there must exist a term 1 such that

n_walks(<1 n) and walks(<1 y) (5)

From the induction hypothesis and eq (5) there exists a term w1 such that

n_walks(x1 w1 n2) and n_walks(w1 n1) (6)

From the eq (5)&(6) and m_one_diamond_one_m there exists a term w2 such that

n_walks(y w2 n2) and walks(w1 w2) (7)

Choose w=w2 From the eq (6)&(7) and definition of n_walk for transitive case we know that n_walks(y w n2) and n_walks(_ w S(n1))

Case 3: (ind2_case) The proof is similar to the previous case except that y and n1 and n2 are interchanged

Case 4 (ind3_case)

We have the induction hypothesis that for fixed n1 and n2

(< y term) n_walks(< y n1) - n_walks(n2) -

∃(w term) (n_walks(y w n2) ^ n_walks(w n1)) (8)

We are given that for fixed y

n_walks(x y S(n1)) and n_walks(S(n2)) (9)

We have to exhibit a w such that y and walk to it in S(n2) and S(n1) walk steps From the definition of n_walks for transitive case and eq (9) there exist terms x1 x2 such that

n_walks(x1 n1) and walks(1 y) (10)

n_walks(x2 n2) and walks(x2) (11)

From the induction hypothesis and eq (10)&(11) there exist w1 such that

n_walks(x1 w1 n2) and n_walks(x2 w1 n1) (12)

From the lemma_one_diamond_one_m and eq (10)&(11)&(12) there exist w2 and w3 such that

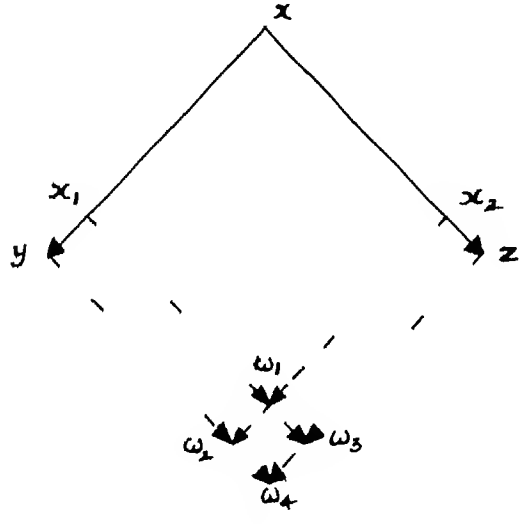
n_walks(y w2 n2) and walks(w1 w2) (13)

n_walks(w3 n1) and walks(w1 w3) (14)

From theorem diamond_walk and eq (13)&(14) there exists a term w4 such that

walks(w2 w4) and walks(w3 w4) (15)

Choose w = w4 Hence from the definition of n_walks for transitive case and eq (13) (14)&(15) we conclude that n_walks(y w S(n2)) and n_walks(w S(n1))



The whole construction of w can be shown in the above diagram

6 Church-Rosser

The key of the proof lies in showing that a b-step can be represented as a walk and a walk can be represented as a series of b steps This actually amounts to saying that b-reduction can be seen as reflexive transitive closure of walks relation We have already seen how a particular reflexive transitive closure relation of walks relation the n_walks relation follows diamond property

6 1 Lemma **b_step_to_walks** :

b_step(u v)->walks(u v)

Proof : From the definition of b_step u can be an abstraction or an application For a particular u and v we have

b step(u v) (1)

Lambda case: Here we have u-lamda(m) From the definition of b_step there exists m1 such that v-lamda(m1) and b_step(m m1) From the definition of walks and inductive hypothesis for m we have walks(u v)

Apply case Here we have u=apply(m1 m2) and apropriate inductive hypotheses for terms m1 and m2 From the definition of b_step three cases arise:

Subcase (a): Here m1 b_steps to n1 So we have

v=apply(n1 m2) and b_steps(m1 n1) (2)

From the inductive hypothesis and eq (2) we have

walks(m1 n1) and walks(m2 m2) (3)

Hence from the definition of walks we have walks(u v)

Subcase (b) _ Here m2 b_steps to n2 This case is similar to the previous one

Subcase (c) _ Here we have m1=lamda(a) and bst(m2 one a v) From the definition of walk we have walks(u v)

6 2 Lemma n_b_steps_to_n_walks :

n_b_step(u v n)->n_walks(u v n)

We know from the previous lemma that a b step can be represented as a walk step Current lemma says that a series of b_steps can be seen as an equal number of walk steps Current lemma can be proved by the induction on the length of reduction chain and from the fact that n_walks and n_b_step are reflexive transitive closures of walks and b_step respectively

6 3 Lemma abs_same_b_steps :

n_b_step(u v n) -> n_b_step(lamda(u) lamda(v) n)

This lemma follow straight forwardly from the definition of n_b_step and induction on the length of reduction

6 4 Lemma b_step_trans_add :

n_b_step(u v n1) -> n_b_step(v w n2) -> n_step(u w add(n1 n2))

This lemma can be proved by induction on either n1 or n2

6 5 Lemma app_add_b_steps :

n_b_step(m1 m3 n1)->n_b_step(m2 m4 n2)->
n_b_step(apply(m1 m2) apply(m3 m4) add(n1 n2))

Proof : This lemma is proved by double induction on n1 n2 We have

n_b_step(m1 m3 n1) and* n_b_step(m2 m4 n2) (1)

Case 1: (Base case) When n1=0 n2=0 we have m1=m3 and m2=m4 from the definition of n_b_step So apply(m1 m2) reduces in zero steps to apply(m3 m4)

Case 2 (Induction step for n1) We have

n_b_step(m1 m3 S(n1)) and n_b_step(m2 m4 n2) (2)

We also have the inductive hypothesis for n1 and n2 We have to prove that

n_b steps(apply(m1 m2) apply(m3 m4) add(S(n1) n2) Hold From the definition of n_walks and eq (2) there exists m1 such that

n_b_step(m1 m1 n1) and b_step(m1 m3) (3)

From the induc tive hypothesis and eq (2)&(3) we know that

n_b_ tep(apply(m1 m2) apply(m1 m4) add(n1 n2)) (4)

From the definition of b_step and eq (3) we have

b_step(apply(m1 m4) apply(m3 m4)) (5)

From the definition of n_b_step and eq (4)&(5) we have

n_b_step(apply(m1 m2) apply(m3 m4) S(add(n1 n2)) (6)

From this it is obvious that n_b_step(apply(m1 m2) apply(m3 m4) add(S(n1) n2))

Case 3&4 _ These two cases are generalisations of the last case

6 6 Lemma walks_to_b_steps :

walks(u v)->∃(n nat)n_b_step(u v n)

This lemma says that a walk step can be represented as a series of b-steps We have

walks(u v) (1)

Proof : The proof is by induction on the structure of the term u

Variable case: Here we have u-variable(v0) From the definition of walk

v=variable(v0) (2)

From the definition of refl_b_step we conclude that the required n is ero

Lambda case We have u=lamda(m) The inductive hypothesis is

(v term) walks(m v)⊃∃(n:nat) n_b_step(m v n) (3)

From the definition of walk there exists m1 such that

walks(m m1) and v=lamda(m1) (4)

From eq (3)&(4) there exists n such that n_b_step(m m1 n) from abs_same_b_steps n_b_step(lamda(m) v n)

Apply case: We have u=apply(m1 m2) We have to exhibit an n such that n_b_step(apply(m1 m2) v n) We have the inductive hypotheses for m1 and m2 that

(v:term) walk (m1 v) ⊃ ∃(n:nat) n_b_step(m1 v n) (5)

(v:term) walks(m2 v) ⊃ ∃(n:nat) n_b_step(m2 v n) (6)

From the definition of walk two cases arise:

Subcase(a) There exist m3 m4 such that

walks(m1 m3) and walks(m2 m4) (7)

From the inductive hypotheses there exist n1 n2 such that

n_b_step(m1 m3 n1) and n_b_step(m2 m4 n2) (8)

From app_add_b_steps add(n1 n2) is the required n

Subcase(b): There exists a a1 m3 m4 such that

m1=lamda(a) m3=lamda(a1) walks(m1 m3)

walks(m2 m4) sbst(m4 one a1 v) (9)

From the inductive hypotheses there exist n1 n2 such that

n_b_step(m1 m3 n1) and n_b_step(m2 m4 n2) (10)

From lemma app add_b_steps and eq (10)

n_b_step(apply(m1 m2) apply(m3 m4) add(n1 n2)) (11)

From the definition of b_step and eq (9)

b_step(apply(m3 m4) v) (12)

From the definition of n_b_step and eq (11) (12) we infer n_b_step(apply(m1 m2) v S(add(n1 n2))) and so S(add(n1 n2)) is the required n

6 7 Lemma n_walks_to_n_b_steps :

n_walks(u v n)->∃(m nat)n_b_step(u v m)

The previous lemma showed how a single wall step can be seen as a number of b steps This lemma says that a walk chain can be seen a a series of b_ teps The proof is by induction on the length of the wall ch ain

6 8 Theorem Church-Rosser

For all terms x y and z reduces(x y)-> reduces(x z)->

∃(w term) reduces(y w) ∧ reduces(z w)

Proof : After unfolding the definition of reduces there e ist n1 n such that

n_b_steps(< y n1) (1)

n_b_steps(< n2) (2)

We have to e<hibit a w such that y and reduce to it From lemma n_b_steps_to_n_walks we conclude that

n_walks(< y n1) (3)

n_walks(x n2) (4)

From the lemma trans_diamond and eq (3)&(4) there must e ist a w such that n_walks(y w n2) and n_walks(z w n1) (5)

From the lemma n_walks_to_n_b_steps there must e ist m1 m2 such that n_b_step(y w m1) and n_b_step(w m2) Hence proved

CHAPTER 4
CONCLUDING REMARKS

In this chapter we briefly mention the arguments for proof checking limitations of the present exercise problems encountered with the system and the future work possible in this direction

The de Bruijn notation is more suitable for computer implementations of λ-calculus With this notation we do away with α-reduction The present study stresses the fact that no essential part of λ calculus is lost by switching to this notation

The proof is presented in a goal directed manner Goal directed proving is the reverse of natural deduction proving In the natural deduction we start with the axioms and reduce them to the goal In the goal directed proving we start with the goal and try to reduce the goal into smaller and smaller subgoals until all the subgoals are reduced to axioms

In the present study we started with the main theorem of Church Rosser as the goal As we tried to reduce this goal into smaller and smaller subgoals we came to the realisation that we needed some more lemmas to proceed further So these lemmas were taken as axioms at that point of time These axioms were taken up later which in turn demanded some more lemmas For example consider the proof of the lemma that a single walk allows the diamond property This needs the sbst_lemma (substitutivity of walks) in the apply case The proof of the sbst_lemma in turn demands a lemma sbst_sbst_lemma in the apply case This in turn demands the lemma incr_sbst_lemma1 which in turn demands incr_incr_lemma In the machine proof these lemmas are developed in the above mentioned order i e sbst_lemma sbst_sbst_lemma incr_sbst_lemma1 and incr_incr_lemma But in the proof presented

in the later these lemmas are presented in the reverse order i e in natural deduction style So the goal directed proof presents a abductive approach since we do not know at the beginning what lemmas are needed for the proof We start with the goal and prove the lemmas as and when needed

One more advantage of the proof checking by machine is clear when the proof becomes very big complex and unwieldly for the humans The proof of subst_ubst_lemma for lambda case clearly shows the advantage of having a proof checker at hand

For the proof of the Church_Rosser not much higher types are needed The proof could have gone through in a first order system also The proof of strong normalisation would have shown more clearly the necessity of an higher order system

One problem with the COC V 4 10 is the parsing stack overflow The tactics theorem prover does not do the complete type checking Only after the complete proof is gone through when we try to save the theorem the complete type checking is done At this point many of the large proofs (involving 20 to 30 existential eliminations) resulted in parsing stack overflows So the bigger proofs have to be broken into small unnatural segments

We have encountered another difficulty with the induction definition facility We wanted a recursive definition of subst operation of type term- nat- term- term This can not be done using the inductive definition facility Inductive definition allows the definition of an altogether new types But subst is an element not a type In order to define the subst we have to define it by the predicate subst :term- nat- term- term- Prop which is the characteristic function of the substitution

Future work possible in this direction is to code the proofs of strong normalisation of typed λ-calculus standardisation theorem and undecidability of the convertibility for pure λ-calculus

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APPENDIX A
HOW TO USE COC

When we execute *constr* command we enter into CAML interactive mode This is indicated by the prompt # In this mode we can verify the proofs contained in constr files by the command

V name of the file

We can enter the Vernacular mode from the CAML mode by typing *go()* The prompt for Vernacular mode is -> All the command of the CAML interactive mode end with double semicolon All the commands of the Vernacular interactive mode end with the full-stop

A typical session of the system in Vernacular mode is shown below The bold text is typed by the user and italic text is the system response We define inductively the type *nat* of natural numbers and the relation *le* of type nat- nat- Prop Then we prove the lemma that if two numbers are in le relation their successors are also in le This lemma is needed in the machine proof for proving the properties of de Bruijn indices

We define nat inductively as

- Inductive Data nat = O : nat | S nat->nat

nat is defined

O is def ned

S s defined

nat_REC is defined

nat_rec s defined

We define relation le inductively as

->Inductive Definition le [n:nat] : nat -> Prop

-> = le_n : (le n n)

```
->     | le_S : (m:nat)(le n m)-> (le n (S m))
le is defined
le_n is defined
le_S is defined
le_rec is defined
le    nat->nat->Prop
le_n    (n nat)(le n n)
le_S    (n nat)(m nat)(le n m)->(le n (S m))
le_rec
(n nat)(x0 nat)(le   n   x0)->(C nat->Prop)(C   n)->((m nat)
      (le n m)/\(C m)->(C (S m)))->(C x0)
```

We start proving the lemma by tactics theorem prover by the command Goal

```
->Goal (i j:nat) (le i j)-> (le (S i) (S j))
```

We make use of intro tactic below

```
->By intros
1 subgoal
  (le (S i) (S j))
   ============================
    H    (le i j)
    j    nat
    i    nat

->Pattern j
1 subgoal
  ((L nat)(le (S i) (S L)) j)
   ============================
    H    (le i j)
    j    nat
```

```
    i    nat

 Pattern j  abstracts the present goal with the variable j
->Resolve H
2 subgoals
  (le (S i) (S i))
  ============================
    H    (le i j)
    j    nat
         nat
subgoal 2 is
  (m nat)(le (S i) (S m))->(le (S i) (S (S m)))

->Resolve le_n
1 subgoal
  (m nat)(le (S i) (S m))->(le (S i) (S (S m)))
  ============================
    H    (le i j)
    j    nat
    i    nat

->By intros
1 subgoal
  (le (S i) (S (S m)))
  ============================
    H0   (le (S i) (S m))
    m    nat
    H    (le    j)
    j    nat
    i    nat

->Resolve le_S
```

```
1 subgoal
  (le (S i) (S m))
  ============================
    H0   (le (S i) (S m))
    m    nat
    H    (le i j)
    j    nat
    i    nat
-/By assumption
goal proved
```

The command Show_proof shows the current completed proof

```
->Show_proof
[i nat][j nat][H (le i j)](H [L nat](le (S i) (S L)) (le_n (S i)))
[m nat][H0 ( le (S i) (S m))](le_S (S i) (S m) H0))
```

Save command saves the theorem so far proved for the rest of the session

```
-/Save le_means_succ_le
le_means_succ_le is defined
```

The command Inspect num prints the last num items

```
-\Inspect 2
le_means_succ_le   (i nat)(j nat)(le i j)->(le (S i) (S j))
le_rec   (n nat)(x0 nat)(le n x0)->( C nat->Prop)
      ( C n)->((m nat)(le n m)/\( C  m)->( C (S m)))->( C x0)
```

That ends the session with the system
The command : *Reset After <ident fier>* resets the system to the state it was in after writing the identifier
The command : *Drop* completes the Vernacular mode and enters into the CAML mode The command *quit()* e its the system from the CAML mode